MATH 590: QUIZ 5 SOLUTIONS

Name:

1. Let $T : V \to W$ be a linear transformation between vector spaces over F. Define the subspaces $\ker(T)$ and $\mathrm{im}(T)$. (4 points)

Solution. $\ker(T) := \{v \in V \mid T(v) = \vec{0}\}$ and $\mathrm{im}(T) := \{w \in W \mid w = T(v), \text{for some } v \in V\}$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (y, -x)$, which is rotation clockwise by 45 degrees. Let $\alpha := \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$ and $\beta := \{f_1 = (2, 1), f_2 = (1, -2)\}$. State and verify the change of basis theorem relating $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$. You may use the fact that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $|A|$ denotes the determinant of A. (6 points)

Solution. We must verify that $[T]_\beta^\beta = [I]_\alpha^\beta \cdot [T]_\alpha^\alpha \cdot [I]_\beta^\alpha$, which is the change of basis theorem.

1. $T(e_1) = (0, -1) = 0 \cdot e_1 + -1 \cdot e_e$ and $T(e_2) = (1, 0) = 1 \cdot e_1 + 0 \cdot e_2$. Thus, $[T]_\alpha^\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2. $T(f_1) = (1, -2) = 0 \cdot f_1 + 1 \cdot f_2$ and $T(f_2) = (-2, -1) = -1 \cdot f_1 + 0 \cdot f_2$. Thus $[T]_\beta^\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

3. $I(f_1) = 2 \cdot e_1 + 1 \cdot e_2$ and $T(f_2) = 1 \cdot e_1 + -2 \cdot e_2$. Thus, $[I]_\beta^\alpha = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$.

4. $[I]_\alpha^\beta = ([I]_\beta^\alpha)^{-1} = -\frac{1}{5} \cdot \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}$.

We now have

$$\begin{aligned} [I]_\alpha^\beta \cdot [T]_\alpha^\alpha \cdot [I]_\beta^\alpha &= -\frac{1}{5} \cdot \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \\ &= -\frac{1}{5} \cdot \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= [T]_\beta^\beta. \end{aligned}$$